The gauge condition in gravitation theory with a background metric

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Abstract. In gravitation theory with a background metric, a gravitational field is described by a two-tensor field. The energy-momentum conservation law imposes a gauge condition on this field.

Gravitation theory in the presence of a background metric remains under consideration. In particular, there are two variants of gauge gravitation theory [9, 10, 12]. The first of them leads to the metric-affine gravitation theory, while the second one (henceforth BMT) deals with a background pseudo-Riemannian metric $g^{\mu\nu}$ and a non-degenerate (1,1)-tensor field $q^\mu{}_\nu$, regarded as a gravitational field. A Lagrangian of BMT is of the form

$$\mathcal{L}_{\rm BMT} = \epsilon\mathcal{L}_q + \mathcal{L}_{\rm AM} + \mathcal{L}_{\rm m}, \tag{1}$$

where $\mathcal{L}_q$ is a Lagrangian of a tensor gravitational field $q^\mu{}_\nu$ in the presence of a background metric $g^{\mu\nu}$, $\mathcal{L}_{\rm AM}$ is a Lagrangian of the metric-affine theory where a metric $g^{\mu\nu}$ is replaced with an effective metric

$$\widetilde{g}^{\mu\nu} = q^\mu{}_\alpha q^\nu{}_\beta g^{\alpha\beta}, \tag{2}$$

and $\mathcal{L}_{\rm m}$ is a matter field Lagrangian depending on an effective metric $\widetilde{g}$ and a general linear connection K (see, e.g., [5]). Note that, strictly speaking, $\widetilde{g}$ (2) is not a metric, but there exists a metric whose coefficients equal $\widetilde{g}^{\mu\nu}$ (2). Therefore, one usually assumes that $\mathcal{L}_q$ depends on q only via an effective metric $\widetilde{g}$.

A glance at the expression (1) shows that the matter field equation in BMT is that of affine-metric theory where a metric g is replaced with an effective metric $\widetilde{g}$. However, gravitational field equations are different because of the term $\epsilon\mathcal{L}_q$. The question is whether solutions of BMT come to solutions of the metric-affine theory if the constant ϵ tends to zero. The answer to this question follows from the energy-momentum conservation law in BMT. We obtain the weak equality

$$\nabla_\lambda t^\lambda_\alpha \approx 0, \tag{3}$$

where ∇_λ is the covariant derivative with respect to the Levi-Civita connection of an effective metric $\widetilde{g}$ and t^λ_α is the metric energy-momentum tensor of the Lagrangian $\mathcal{L}_q$

with respect to $\widetilde{g}$. The equality (3) holds for any solution q of field equations of BMT. This equality is defined only by the Lagrangian $\mathcal{L}_q$, and is independent of other fields and the constant ϵ. Therefore, it can be regarded as a gauge condition on solutions of BMT.

Recall that, in gauge theory on a fibre bundle $Y \to X$ coordinated by (x^λ, y^i), gauge transformations are defined as bundle automorphisms of $Y \to X$ (see [3, 6, 11] for a survey). Their infinitesimal generators are projectable vector fields

$$u = u^\lambda(x^\mu)\partial_\lambda + u^i(x^\mu, y^j)\partial_i \tag{4}$$

on a fibre bundle $Y \to X$. We are concerned with a first order Lagrangian field theory on Y. Its configuration space is the first order jet manifold J^1Y of $Y \to X$ coordinated by $(x^\lambda, y^i, y^i_\lambda)$. A first order Lagrangian is defined as a density

$$L = \mathcal{L}(x^\lambda, y^i, y^i_\lambda)dx^n, \qquad n = \dim X, \tag{5}$$

on J^1Y. A Lagrangian L is invariant under a one-parameter group of gauge transformations generated by a vector field u (4) iff its Lie derivative

$$\mathbf{L}_{J^1u}L = J^1u\rfloor dL + d(J^1u\rfloor L) \tag{6}$$

along the jet prolongation J^1u of u onto J^1Y vanishes. By virtue of the well-known first variation formula, the Lie derivative (6) admits the canonical decomposition

$$\mathbf{L}_{J^1u}L = (u^i - y^i_\mu u^\mu)\delta_i\mathcal{L}d^nx - d_\lambda\mathfrak{T}^\lambda d^nx, \tag{7}$$

where $d_\lambda = \partial_\lambda + y^i_\lambda\partial_i + y_{\lambda\mu}\partial^\mu_i$ denotes the total derivative,

$$\delta_i\mathcal{L} = (\partial_i - d_\alpha\partial^\lambda_i)\mathcal{L} \tag{8}$$

are variational derivatives, and

$$\mathfrak{T}^\lambda = (u^\mu y^i_\mu - u^i)\partial^\lambda_i\mathcal{L} - u^\lambda\mathcal{L} \tag{9}$$

is a current along the vector field u. If the Lie derivative $\mathbf{L}_{J^1u}L$ vanishes, the first variation formula on the shell $\delta_i\mathcal{L} = 0$ leads to the weak conservation law $d_\lambda\mathfrak{T}^\lambda_u \approx 0$. In particular, if $u = \tau^\lambda\partial_\lambda + u^i\partial_i$ is a lift onto Y of a vector field $\tau = \tau^\lambda\partial_\lambda$ on X (i.e., u^i is linear in τ^λ and their derivatives), then $\mathfrak{T}^\lambda$ is an energy-momentum current [1, 3, 4, 7]. Note that different lifts onto Y of a vector field τ on X lead to distinct energy-momentum currents whose difference is a Noether current. Gravitation theory deals with fibre bundles over X which admit the canonical lift of any vector field on X. This lift is a generator of general covariant transformations [3, 8].

Let X be a 4-dimensional oriented smooth manifold satisfying the well-known topological conditions of the existence of a pseudo-Riemannian metric. Let LX denote the fiber bundle of oriented frames in TX. It is a principal bundle with the structure group $GL_4 = GL^+(4,\mathbb{R})$. A pseudo-Riemannian metric g on X is defined as a global section of the quotient bundle

$$\Sigma = LX/SO(1,3) \to X. \tag{10}$$

This bundle is identified with an open subbundle of the tensor bundle $\overset{2}{\vee} TX$. Therefore, Σ can be equipped with coordinates $(\xi^\lambda, \sigma^{\mu\nu})$, and g is represented by tensor fields $g^{\mu\nu}$ or $g_{\mu\nu}$.

Principal connections K on the frame bundle LX are linear connections

$$K = dx^\lambda \otimes (\partial_\lambda + K_\lambda{}^\mu{}_\nu \dot{x}^\nu \dot{\partial}_\mu) \tag{11}$$

on the tangent bundle TX and other tensor bundles over X. They are represented by sections of the quotient bundle

$$C = J^1LX/GL_4 \to X, \tag{12}$$

where J^1LX is the first order jet manifold of the frame bundle $LX \to X$ [3, 6, 11]. The bundle of connections C is equipped with the coordinates $(x^\lambda, k_\lambda{}^\nu{}_\alpha)$ such that the coordinates $k_\lambda{}^\nu{}_\alpha \circ K = K_\lambda{}^\nu{}_\alpha$ of any section K are the coefficients of corresponding linear connection (11).

A tensor gravitational field q is defined as a section of the group bundle $Q \to X$ associated with LX. Its typical fiber is the group GL_4 which acts on itself by the adjoint representation. The group bundle Q as a subbundle of the tensor bundle $TX \otimes T^*X$ is equipped with the coordinates $(x^\lambda, q^\lambda{}_\mu)$. The canonical left action Q on any bundle associated with LX is given. In particular, its action on the quotient bundle Σ (10) takes the form

$$\rho : Q \times \Sigma \to \Sigma, \qquad \rho : (q^\lambda{}_\mu, \sigma^{\mu\nu}) \mapsto \widetilde{\sigma}^{\mu\nu} = q^\mu{}_\alpha q^\nu{}_\beta \sigma^{\alpha\beta}.$$

Since the Lagrangian (1) of BMT depends on q only via an effective metric, let us further replace variables q with the variables $\widetilde{\sigma}$. Then, the configuration space of BMT is the jet manifold J^1Y of the product $Y = \Sigma \times C \times Z$, where $Z \to X$ is a fibre bundle of matter fields. Relative to coordinates $(\widetilde{\sigma}^{\mu\nu}, k_\lambda{}^\alpha{}_\beta, \phi)$ on Y, the Lagrangian (1) reads

$$\mathcal{L}_{\rm BMT} = \epsilon \mathcal{L}_q(\sigma, \widetilde{\sigma}) + \mathcal{L}_{\rm AM}(\widetilde{\sigma}, R) + \mathcal{L}_m(\phi, \widetilde{\sigma}, k), \tag{13}$$

where the metric-affine Lagrangian $\mathcal{L}_{AM}$ is expressed into components of the curvature tensor

$$R_{\lambda\mu}{}^\alpha{}_\beta = k_{\lambda\mu}{}^\alpha{}_\beta - k_{\mu\lambda}{}^\alpha{}_\beta + k_\lambda{}^\varepsilon{}_\beta k_\mu{}^\alpha{}_\varepsilon - k_\mu{}^\varepsilon{}_\beta k_\lambda{}^\alpha{}_\varepsilon$$

contracted by means of the effective metric $\widetilde{\sigma}$. The corresponding field equations read

$$\delta_{\mu\nu}(\epsilon\mathcal{L}_q + \mathcal{L}_{\rm AM} + \mathcal{L}_m) = 0, \tag{14}$$
$$\delta^\lambda{}_\alpha{}^\beta(\mathcal{L}_{\rm AM} + \mathcal{L}_m) = 0, \tag{15}$$
$$\delta_\phi\mathcal{L}_m = 0.$$

where $\delta_{\mu\nu}$, $\delta^\lambda{}_\alpha{}^\beta$ and δ_ϕ are variational derivatives (8) wit respect to $\widetilde{\sigma}^{\mu\nu}$, $k_\lambda{}^\alpha{}_\beta$ and ϕ.

Let $\tau = \tau^\lambda\partial_\lambda$ be a vector field on X. Its canonical lift onto the fibre bundle $\Sigma \times C$ reads

$$\widetilde{\tau} = \tau^\lambda\partial_\lambda + (\partial_\nu\tau^\alpha k_\mu{}^\nu{}_\beta - \partial_\beta\tau^\nu k_\mu{}^\alpha{}_\nu - \partial_\mu\tau^\nu k_\nu{}^\alpha{}_\beta + \partial_{\mu\beta}\tau^\alpha)\frac{\partial}{\partial k_\mu{}^\alpha{}_\beta} \tag{16}$$
$$+(\partial_\varepsilon\tau^\mu\widetilde{\sigma}^{\varepsilon\nu} + \partial_\varepsilon\tau^\nu\widetilde{\sigma}^{\mu\varepsilon})\frac{\partial}{\partial\widetilde{\sigma}^{\mu\nu}} = \tau^\lambda\partial_\lambda + u_\mu{}^\alpha{}_\beta\partial^\mu{}_\alpha{}^\beta + u^{\mu\nu}\partial_{\mu\nu}.$$

It is a generator of general covariant transformations of the fiber bundle $\Sigma \times C$. Let us apply the first variation formula (7) to the Lie derivative $\mathbf{L}_{J^1\widetilde{\tau}}L_{\rm MA}$. Since the Lagrangian $L_{\rm MA}$ is invariant under general covariant transformations, we obtain the equality

$$0 = (u^{\mu\nu} - \sigma^{\mu\nu}_\varepsilon\tau^\varepsilon)\delta_{\mu\nu}\mathcal{L}_{\rm MA} + (u_\lambda{}^\alpha{}_\beta - k_{\varepsilon\lambda}{}^\alpha{}_\beta\tau^\varepsilon)\delta^\lambda{}_\alpha{}^\beta\mathcal{L}_{\rm MA} - d_\lambda\mathfrak{T}^\lambda_{\rm MA}, \tag{17}$$

where $\mathfrak{T}_{\rm MA}$ is the energy-momentum current of the metric-affine gravitation theory. It reads

$$\mathfrak{T}^\lambda_{\rm MA} = 2\widetilde{\sigma}^{\lambda\mu}\tau^\alpha\delta_{\alpha\mu}\mathcal{L}_{\rm MA} + T(\delta^\lambda{}_\alpha{}^\beta\mathcal{L}_{\rm MA}) + d_\mu U^{\mu\lambda}_{\rm MA}, \tag{18}$$

where $T(.)$ are terms linear in the variational derivatives $\delta^\lambda{}_\alpha{}^\beta\mathcal{L}_{\rm MA}$ and $U^{\mu\lambda}_{\rm MA}$ is the generalized Komar superpotential [2, 3].

For the sake of simplicity, let us assume that there exists a domain $N \subset X$ where $L_m = 0$, and let us consider the equality (17) on N. The field equations (14) – (15) on N take the form

$$\delta_{\mu\nu}(\epsilon\mathcal{L}_q + \mathcal{L}_{\rm AM}) = 0, \tag{19}$$
$$\delta^\lambda{}_\alpha{}^\beta\mathcal{L}_{\rm AM} = 0, \tag{20}$$

and the energy current (18) reads

$$\mathfrak{T}^\lambda_{\rm MA} = 2\widetilde{\sigma}^{\lambda\mu}\tau^\alpha\delta_{\alpha\mu}\mathcal{L}_{\rm MA} + d_\mu U^{\mu\lambda}_{\rm MA}. \tag{21}$$

Substituting (19) and (21) into (17), we obtain the weak equality

$$0 \approx -(2\partial_\lambda\tau^\alpha\sigma^{\lambda\mu} - \sigma^{\alpha\mu}_\lambda\tau^\lambda)\delta_{\alpha\mu}\mathcal{L}_q + d_\lambda(2\widetilde{\sigma}^{\lambda\mu}\tau^\alpha\delta_{\alpha\mu}\mathcal{L}_q). \tag{22}$$

A simple computation brings this equality into the desired form (3) where

$$t^\lambda_\alpha = 2\widetilde{g}^{\lambda\nu}\sqrt{-\widetilde{g}}\delta_{\nu\alpha}\mathcal{L}_q.$$

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